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A NOTE ON FRAGMENTS OF INFINITE GRAPHS

by

H. A. JUNG

Technische Universität Berlin
Fachbereich Mathematik, 1000 Berlin 12

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Dedicated to Prof. K. Wagner on his 70th birthday

Results involving automorphisms and fragments of infinite graphs are proved. In particular for a given fragment C and a vertex-transitive subgroup G of the automorphism group of a connected graph there exists $\sigma \in G$ such that $\sigma[C] \subset C$. This proves the countable case of a conjecture of L. Babai and M. E. Watkins concerning graphs allowing a vertex-transitive torsion group action.

1. Introduction

Given a graph X, let $\varkappa_\infty$ denote the least cardinality of a vertex set which separates two infinite subsets of the vertex set $V(X)$ of X. In this note graphs X with finite $\varkappa_\infty$ are considered. If S separates the infinite sets C_1, C_2 where $|S| = \varkappa_\infty$ and $V(X) = C_1 \cup S \cup C_2$, then C_1, C_2 are called *fragments* of X.

L. Babai and M. E. Watkins [1] showed that for locally finite graphs X with $0 < \varkappa_\infty < \infty$ any torsion subgroup of the automorphism group Aut X of X has infinitely many orbits with respect to the action on $V(X)$. They also conjectured that graphs Y with prescribed connectivity $\varkappa > 0$ and allowing a vertex-transitive torsion group of automorphisms cannot have arbitrarily large valence. Note that in the case $\varkappa < \infty$ either $\varkappa_\infty < \infty$ or some $v \in V(Y)$ has finite valence.

In fact, the following is a consequence of Theorem 1 below.

Corollary. *If Y allows a vertex-transitive torsion group of automorphisms and $0 < \varkappa < \infty$, then Y is locally finite.*

The *boundary* ∂C of $C \subseteqq V(X)$ is the set of vertices in $V(X) - C$ which are joined to some vertex in C.

Theorem 1. *Let $\varkappa_\infty < \infty$ and let G be a vertex-transitive subgroup of* Aut X. *Then for each fragment C of X there exist $\sigma \in G$ such that $\sigma[C \cup \partial C] \subseteqq C$.*

Obviously an automorphism σ as described in Theorem 1 has infinite order provided X is connected.

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2. Rigid neighbours of fragments

Using the abbreviation $\bar{S}=V(X)-(S\cup\partial S)$ one may define (cf. [3]):

$$\varkappa = \varkappa(X) = \min(|\partial S|:\ S \text{ and } \bar{S} \text{ non-empty})$$

and $\varkappa_\infty=\varkappa_\infty(X)=\min\{|\partial S|:\ S \text{ and } \bar{S} \text{ infinite}\}$.

We call v a *rigid* neighbour of the fragment C if $v\in\partial D$ for all fragments $D\subseteqq C$. The set of rigid neighbours of C is denoted by $\partial^* C$.

Lemma 2.1. *Let $\varkappa_\infty<\infty$ and let C be a fragment of X. If D is a fragment and $D\subseteqq C$ then $\partial^* C\subseteqq\partial^* D$. If $\partial^* D'=\partial^* C$ for all fragments $D'\subseteqq C$ then $C\cap\partial^* D=\emptyset$ for all fragments D of X.*

Proof. The first assertion is an immediate consequence of the definitions. Suppose $\partial^* D=\partial^* C$ for each fragment D of X such that $D\subseteqq C$.

Let D be a fragment of X. By Corollary 2A in [3], $D\cap C$ or $D\cap\bar{C}$ is a fragment of X (with respect to $\varkappa_\infty$). Hence

$$\partial^* D \subseteqq \partial^*(D\cap C) = \partial^* C$$

or

$$\partial^* D \subseteqq \partial(D\cap\bar{C}) \subseteqq \bar{C}\cup\partial D = \bar{C}\cup\partial C. \quad \blacksquare$$

As usual $d(s, t)$ denotes the number of edges on a shortest path joining the vertices s, t; further

$$\operatorname{diam} S = \max\left(d(s_1, s_2):\ s_1, s_2\in S\right)$$

and

$$d(S,T) = \min\left(d(s,t):\ s\in S,\ t\in T\right)$$

for finite sets S, $T\subseteqq V(X)$.

Lemma 2.2. *Let $0<\varkappa_\infty<\infty$ and let C be a fragment of X such that $\partial^* D=\emptyset$ for all fragments $D\subseteqq C$. Then for each positive integer n there is a fragment C_1 of X such that $C_1\cup\partial C_1\subseteqq C$ and $d(\partial C_1, \partial C)\geqq n$.*

Proof. Let $\partial C=\{v_1, v_2, \dots, v_k\}$. One first constructs fragments $C\supseteqq D_1\supseteqq D_2\supseteqq\dots\supseteqq D_k$ such that $v_i\notin\partial D_i$ $(i=1, 2, \dots, k)$. Setting $C=D_0$, assume D_i has been constructed and $0\leqq i<k$. Since $v_{i+1}\notin\partial^* D_i$ one can find a fragment D_{i+1} such that $D_{i+1}\subseteqq D_i$ and $v_{i+1}\notin\partial D_{i+1}$. Clearly $D_{i+1}\cup\partial D_{i+1}\subseteqq D_i\cup\partial D_i\subseteqq C\cup\partial C$. Hence $D_k\cup\partial D_k\subseteqq C$.

Using the construction of the previous paragraph one can construct fragments $C=C^{(0)}\supseteqq C^{(1)}=D_k\supseteqq C^{(2)}\supseteqq\dots\supseteqq C^{(n)}$ such that $C^{(i)}\cup\partial C^{(i)}\subseteqq C^{(i-1)}$ for $0<i\leqq n$. Clearly $d(\partial C^{(n)}, \partial C)\geqq n$. $\blacksquare$

3. Automorphisms and Fragments

We call a fragment C of X and $\sigma\in\operatorname{Aut} X$ *compatible* if $\sigma[\partial C]\subseteqq C$.

Lemma 3.1. *Let $0<\varkappa_\infty<\infty$ and let C, σ be compatible. Then either*
(i) $\bar{C}\cup\partial C\subseteqq\sigma[\bar{C}]$ *and* $\sigma[C\cup\partial C]\subseteqq C$ *or*
(ii) $\bar{C}\cup\partial C\subseteqq\sigma[C]$ *and* $\sigma[\bar{C}\cup\partial C]\subseteqq C$.

Proof. Since $\bar{C}\cup\partial C$ induces a connected subgraph of $X-\sigma[\partial C]$ one has $\bar{C}\cup\partial C\subseteqq\sigma[\bar{C}]$ or $\bar{C}\cup\partial C\subseteqq\sigma[C]$; also, by complementation, $\sigma[C\cup\partial C]\subseteqq C$ or $\sigma[\bar{C}\cup\partial C]\subseteqq C$ respectively. ∎

Lemma 3.2. *Let $0<\varkappa_\infty<\infty$, and let C be a fragment of X. Further let G be a vertex-transitive group of automorphisms of X. Then $\partial^* C=\emptyset$, and there exists $\sigma\in G$ such that C, σ are compatible.*

Proof. One can find a fragment $C_1\subseteqq C$ such that $\partial^* D=\partial^* C_1$ for all fragments $D\subseteqq C_1$. For $x\in\partial^* C_1$ one could pick $\sigma\in G$ such that $\sigma(x)\in C_1$, and one would have $\sigma(x)\in\sigma[\partial^* C_1]=\partial^*\sigma[C_1]$ contrary to Lemma 2.1. Hence $\emptyset=\partial^* C_1\supseteqq\partial^* C$. By Lemma 2.2 there exists a fragment $C_2\subseteqq C$ such that $C_2\cup\partial C_2\subseteqq C$ and $d(\partial C_2, \partial C)\geqq\operatorname{diam}\partial C$. For any $\sigma\in\operatorname{Aut} X$ such that $\sigma[\partial C]\cap C_2\neq\emptyset$ one has $\sigma[\partial C]\subseteqq C$. ∎

Theorem 1 is an immediate consequence of Lemma 3.2 and the following result.

Proposition 3.3. *Let $0<\varkappa_\infty<\infty$ and let C be a fragment of X. If C, σ are compatible and $\bar{C}$, τ are compatible then $\varphi[C\cup\partial C]\subseteqq C$ for some $\varphi\in\{\sigma, \tau^{-1}, \sigma^{-1}\tau^{-1}\}$.*

Proof. According to Lemma 3.1 one may assume $\bar{C}\cup\partial C\subseteqq\sigma[C]$ and $\sigma[\bar{C}\cup\partial C]\subseteqq C$. Also, by Lemma 3.1, either $C\cup\partial C\subseteqq\tau[C]$ or $C\cup\partial C\subseteqq\tau[\bar{C}]$. In the latter case $\tau^{-1}[C\cup\partial C]\subseteqq\bar{C}\subseteqq\sigma[C]$. ∎

It must be pointed out that the proof of theorem 1 — with the exception of the arguments involving rigid neighbours — is very similar to the proof of Theorem 2 in [1].

R. Halin [2] called $\sigma\in\operatorname{Aut} X$ of type 1 if $\sigma(F)=F$ for some finite non-empty $F\subseteqq V(X)$. Clearly, an automorphism σ as described in Theorem 1 is not of type 1. Theorem 2 and Theorem 3 deal with automorphisms of type 1.

Theorem 2. *Let $\varkappa_\infty<\infty$ and $\sigma^n[C]=C$ for some fragment C of X and some positive integer n. Then there exists some fragment $D\subseteqq C$ such that $\sigma^n[D]=D$ and the sets $D, \sigma[D], \ldots, \sigma^{n-1}[D], \partial D\cup\sigma[\partial D]\cup\ldots\cup\sigma^{n-1}[\partial D]$ are pairwise disjoint or equal.*

Proof. Put $C^{(0)}=C$ and $C^{(1)}=\bar{C}$ and construct a sequence $e_0, e_1, \ldots, e_{n-1}$ in $\{0, 1\}$ such that $C^{(e_0)}\cap\sigma[C^{(e_1)}]\cap\ldots\cap\sigma^i[C^{(e_i)}]=D_i$ is a fragment of X $(i=0, 1, \ldots, n-1)$. If D_i is constructed and $i<n-1$ then, by Corollary 2A in [3], the set $D_i\cap\sigma^{i+1}[C]$ or $\overline{D_i\cap\sigma^{i+1}[C]}=D_i\cap\sigma^{i+1}[\bar{C}]$ is a fragment. Therefore one can pick $e_{i+1}\in\{0, 1\}$ such that $D_{i+1}=D_i\cap\sigma^{i+1}[C^{(e_{i+1})}]$ is a fragment. It will be shown that $D=D_{n-1}$ has the desired properties. Clearly $\sigma^n[D]=D$ since $\sigma^n[\bar{C}]=\bar{C}$. For $j\in\mathbf{Z}$ the set $\sigma^j[D]$ has the form $C^{(f_0)}\cap\sigma[C^{(f_1)}]\cap\ldots\cap\sigma^{n-1}[C^{(f_{n-1})}]$ where $f_1, f_2, \ldots, f_{n-1}\in\{0, 1\}$. A vertex $x\in\sigma^j[D]\cap D$ belongs to

$$\sigma^i[C^{(f_i)}]\cap\sigma^i[C^{(e_i)}] \quad \text{for} \quad 0\leqq i\leqq n-1.$$

If such an x exists, one has $e_i=f_i$ for all i. Hence $\sigma^j[D]=D$. Therefore $\sigma^j[D]\cap\cap\sigma^k[D]\neq\emptyset$ implies $\sigma^j[D]=\sigma^k[D]$.

Since $\partial D\subseteqq\partial C\cup\sigma[\partial C]\cup\ldots\cup\sigma^{n-1}[\partial C]$, any vertex y in $\partial D\cap\sigma^j[D]$ belongs to some $\sigma^k[\partial C]\cap\sigma^j[D]$ where $0\leqq k\leqq n-1$. But $\sigma^j[D]\subseteqq\sigma^k[C^{(f_k)}]$, and hence no such y exists. Therefore $\sigma^e[\partial D]\cap\sigma^j[D]=\emptyset$ for all e and j. ∎

The following related result is concerned with the locally finite case.

Theorem 3. *Let* $0<\varkappa_\infty<\infty$ *and let* $\sigma\in\operatorname{Aut} X$ *be of type* 1. *Then for each fragment* C *of* X *there exist an integer* $n>0$ *and a fragment* $D\subseteqq C$ *such that* $\sigma^n[D]=D$ *and any two members of the sequence* $D, \sigma[D], \ldots, \sigma^{n-1}[D]$ *and* $\partial D\cup\sigma[\partial D]\cup\ldots\cup\sigma^{n-1}[\partial D]$ *are disjoint.*

Proof. By definition, $\sigma[F]=F$ for some finite $F\subseteqq V(X)$, $F\neq\emptyset$. The set $S=\bigcup_{i\in\mathbf{Z}}\sigma^i[\partial C]$ also is finite since $d(x, F)\leqq\max\big(d(y, F)\colon y\in\partial C\big)$ for all $x\in S$. Clearly $\sigma[S]=S$. There are only finitely many fragments whose boundary is contained in S and hence one can find a fragment D which is minimal subject to the conditions $D\subseteqq C$ and $\partial D\subseteqq S$. It will be shown that such a D has the desired property.

Let $\sigma^i[D\cup\partial D]\cap D\neq\emptyset$. Then $D\cap\sigma^i[D]$ or $D\cap\sigma^i[\bar{D}]$ is a fragment D_1 and $\partial D_1\subseteqq\partial D\cup\sigma^i[\partial D]\subseteqq S$. Hence $D_1=D$ which implies $\sigma^i[D]\supseteqq D$. Therefore $\sigma^i[D]=D$.

In general, $\sigma^i[D\cup\partial D]\cap\sigma^j[D]\neq\emptyset$ implies $\sigma^{i-j}[D\cup\partial D]\cap D\neq\emptyset$ and hence $\sigma^{i-j}[D]=D$ by the preceding argument. This shows that the sequence $D, \sigma[D], \ldots$ is periodic. Moreover the claim holds with the minimum positive integer n such that $\sigma^n[D]=D$. ∎

References

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